# SHEAF SPACES AS KAN EXTENSIONS

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Abstract. We motivate the explicit construction of the sheaf space as a gluing of stalks by first introducing its universal property, which we will consider as an example of a more general class of constructions, that being Kan extensions on presheaf categories. We will use abstract categorical results to prove the equivalence between the category of sheaves and étale spaces and deduce some useful consequences.

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## 1. Motivation

Let $X$ be a topological space. Given a bundle $p\colon E \to X$, we can define a presheaf on X by assigning to each open set $U \subseteq X$ the set of sections

$$\Gamma_p(U) := \{s\colon U \to E \text{ continuous function} \mid ps = 1_X\}.$$

We can observe the functoriality of this construction as follows: first, we notice that a bundle over $X$ is just an element of the slice category $\mathsf{Top}/X$. The slice category admits an obvious embedding from the category of open sets of $X$, which we denote as $\mathsf{Op}(X)$, by sending any open $U \subseteq X$ to the open inclusion map $U \hookrightarrow X$. Call this embedding $J\colon \mathsf{Op}(X) \to \mathsf{Top}/X$. Now, we see that

$$\Gamma_p = {}^{\mathsf{Top}}/_X\Big(J(-), E \xrightarrow{p} X\Big)$$

gives the correct assignment on both open sets and inclusion relations, so it is indeed a presheaf. In addition, the assignment $p \mapsto \Gamma_p$ is functorial from $\mathsf{Top}/X \to \mathsf{Psh}\,X$, as it is the restricted Yoneda embedding

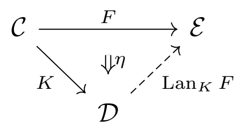
$$\Gamma = {}^{\mathsf{Top}}/_X(J,-) = \Big({}^{\mathsf{Top}}/_X \overset{\text{よ}}{\hookrightarrow} \mathsf{Set}^{({}^{\mathsf{Top}}/_X)^{\mathsf{op}}} \xrightarrow{J^*} \mathsf{Set}^{\mathsf{Op}(X)^{\mathsf{op}}} = \mathsf{Psh}\,X\Big). \tag{1.1}$$

A natural question to ask is whether we can find some kind of inverse functor which encodes any presheaf as the section of some bundle. It turns out that this is partially possible: we can find an adjoint to $\Gamma$ which associates to each presheaf an *étale space*, and this adjunction restricts to an equivalence on the full subcategory of sheaves. Typically (in an algebraic geometry textbook, for instance), this adjoint is defined manually by gluing together stalks. In this document we will instead introduce this adjunction using the general framework of Kan extensions, and then show that the usual construction does satisfy the correct universal property.

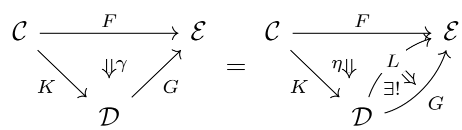
## 2. Kan Extensions

We begin with a quick review of Kan extensions. For a detailed treatment, see Chapter 1 of [Rie14].

**Definition 2.1.** Let $F\colon \mathcal{C} \to \mathcal{E}$ and $K\colon \mathcal{C} \to \mathcal{D}$ be functors. A *left Kan extension* of $F$ along $K$ is a functor $\operatorname{Lan}_K F\colon \mathcal{D} \to \mathcal{E}$ with a natural transformation $\eta\colon F \Rightarrow \operatorname{Lan}_K F \circ K$




such that for any functor $G\colon \mathcal{D} \to \mathcal{E}$, natural transformations $\gamma\colon F \Rightarrow GK$ factor uniquely through $\eta$ as follows: (denoting $L := \operatorname{Lan}_K F$ for clarity)




A functor $H\colon \mathcal{E} \to \mathcal{F}$ *preserves* the Kan extension if the composite $(H \operatorname{Lan}_K F, H\eta)$ is a Kan extension of $K$ along $HF$. Suppose $\mathcal{E}$ is locally small. A Kan extension in this setting is *pointwise* if it is preserved by all (contravariant) representable functors $\mathcal{E}(-, x)\colon \mathcal{E} \to \mathsf{Set}^{\mathsf{op}}$ for $x \in \mathcal{E}$.

*Remark* 2.2. Dually, one can define right Kan extensions (see [Rie14], Definition 1.1.1), but we will not be using them for the rest of this exposition.

**Proposition 2.3.** A left Kan extension is pointwise if and only if it can be computed by the formula

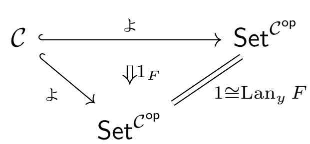
$$\operatorname{Lan}_K F(d) = \int^{c \in \mathcal{C}} \mathcal{D}(Kc, d) \cdot Fc$$

for all $d \in \mathcal{D}$. In particular, this coend always defines a left Kan extension when it exists (for instance, when $\mathcal{E}$ is cocomplete). Equivalently, the coend is described as the colimit

$$\operatorname{Lan}_K F(d) = \operatorname{colim}(K \downarrow d \xrightarrow{\Pi^d} \mathcal{C} \xrightarrow{F} \mathcal{E}).$$

*Proof.* See [Rie14], Theorem 1.2.1 and 1.3.5. □

**Theorem 2.4** (density)**.** For a small category $\mathcal{C}$, the identity functor (and natural transformation) defines the Kan extension of the Yoneda embedding along itself:

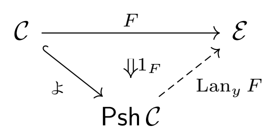



In particular, for any $P \in \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$, we have an expression of $P$ as a coend/colimit

$$P \cong \int^{c \in \mathcal{C}} Pc \cdot \mathcal{C}(-, c).$$

*Proof.* See [Rie16], Theorem 6.5.7 and 6.5.8. □

**Corollary 2.5.** Let $\mathcal{C}$ be small, and let $\mathcal{E}$ be locally small and cocomplete. Then, the pointwise left Kan extension exists and defines a genuine extension of $F$ along $y$. This means that that $\operatorname{Lan}_y F \circ y = F$, and the universal natural transformation is the identity:

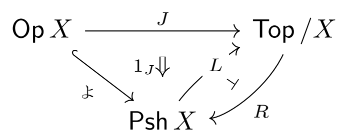



In addition, $\operatorname{Lan}_y F$ admits a right adjoint, defined by $R := \mathcal{E}(F, -)$.

*Proof.* See [Rie16], Remark 6.5.9. □

## 3. Presheaves and Bundles

Now, we are ready to construct the promised adjunction between presheaves and bundles. Applying Proposition 2.5 to the previously defined embedding $J$ that sends an open set of $X$ to the inclusion map regarded as a bundle, we have the following diagram




(3.1)

where $L$ is a pointwise left Kan extension of $J$ along $y$, sending a presheaf $P$ to

$$\begin{aligned} L(P) &= \int^{U \in \mathsf{Op}\,X} \mathsf{Psh}\,X(yU, P) \cdot (U \hookrightarrow X) \\ &\cong \int^{U \in \mathsf{Op}\,X} PU \cdot (U \hookrightarrow X) \end{aligned} \tag{3.2}$$

by the Yoneda lemma. In addition, we observe that $R$ as defined in Proposition 2.5 is exactly the functor $\Gamma = {}^{\mathsf{Top}}\!/_X(J, -)$ from equation (1.1) as a formal consequence of the universal property of the Kan extension. Let's now give an explicit construction for the coend in (3.2).

### 3.1. Construction of the Sheaf Space.

We will use a general fact about slice categories to help us construct colimits in the bundle category $\mathsf{Top}\,/X$.

**Proposition 3.3.** For any category $\mathcal{C}$ and any $c \in \mathcal{C}$, the forgetful functor $\Pi\colon \mathcal{C}/c \to \mathcal{C}$ strictly creates colimits and connected limits.

*Proof.* See [Rie16], Proposition 3.3.8. □

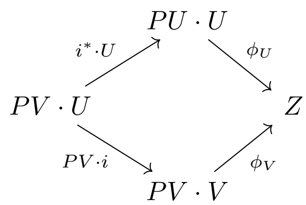
Hence, to construct the bundle $L(P)$, we can just consider the total space (which we call the *sheaf space* of $P$)

$$\Lambda_P := \Pi L(P) = \int^{U \in \mathsf{Op}\,X} PU \cdot U \in \mathsf{Top} \tag{3.4}$$

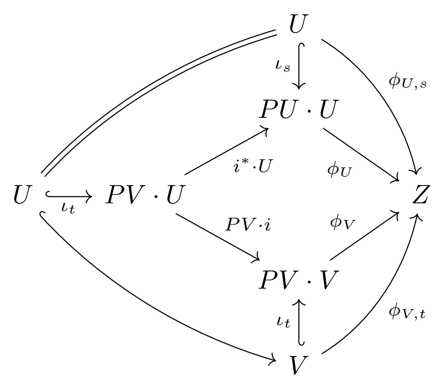
without worrying about the bundle map, which will be strictly created. Let's unpack the universal property of this coend. For any topological space $Z$, a map

$$\phi\colon \left(\int^{U \in \mathsf{Op}\,X} PU \cdot\right) U \to Z$$

is the same as a cowedge, or a collection of morphisms $\{\phi_U : PU \cdot U \to Z\}_{U \in \mathsf{Op}\, X}$ such that for all $i : U \hookrightarrow V \in \mathsf{Op}\, X$, the diagram

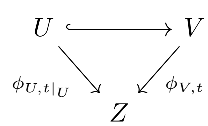



commutes. We can use the universal property of each $PV \cdot U$ to simplify this further. Recall that $PV \cdot U$ is the coproduct of copies of $U$ indexed by sections $t \in PV$. Hence, the map $\phi_U : PU \cdot U \to Z$ is the same as a set of maps $\{\phi_{U,s} : U \to Z\}_{s \in PU}$, and the commutativity condition becomes the conditions that $t|_U = s$ and the outer square in the diagram




for all $s \in PU$ and all $t \in PV$ such that $t|_U = s$. To summarize, we have the following proposition:

**Proposition 3.5.** A cowedge from $P(-) \cdot (=)$ to $Z$ is given by a set of morphisms $\{\phi_{U,s} : U \to Z\}_{U \in \mathsf{Op}\, X, s \in PU}$ such that for all $V \in \mathsf{Op}\, X$, all $t \in PV$ and all $U \subseteq V \in \mathsf{Op}\, X$, the diagram




(3.6)

commutes. A coend is an object with a set of initial such maps.

By the cocompleteness of $\mathsf{Top}$, our desired coend exists, but we will give an explicit construction that is useful in an algebraic geometry setting. We will need the definition of the stalk, which we recall now.

**Definition 3.7.** Let $P$ be a presheaf on $X$. The *stalk* of $P$ at a point $x \in X$ is the set

$$P_x := \operatorname*{colim}_{x \in U} PU,$$

which can be described explicitly as the quotient

$$P_x \cong \{(U, s) \mid U \in \mathsf{Op}\, X \text{ such that } x \in U,\ s \in PU\} / \sim$$

with $(U, s) \sim (V, t)$ if and only if there exists an open set $W \subseteq U \cap V$ containing $x$ such that $s$ and $t$ agree on $W$. We will denote such an equivalence class by $[U, s]_x$, and call it the *germ* of $s$ at $x$. This description comes from the fact that the indexing category of the colimit is filtered (see [MM94], beginning of chapter II.5 for details).

We take the underlying set of $\Lambda_P$ to be the disjoint union

$$\Lambda_P := \coprod_{x \in X} P_x.$$

For any open set $U \subseteq X$ and any section $s \in PU$, we define a map $\dot{s} \colon U \to \Lambda_P$ by

$$\dot{s}(x) := [U, s]_x \in P_x \text{ for all } x \in X,$$

and we topologize $\Lambda_P$ by taking the finest topology with respect to $\dot{s}$ for all opens $U \subseteq X$ and all sections $s \in PU$.

**Lemma 3.8.** A base of topology for $\Lambda_P$ is given by

$$\mathcal{B}_P := \{\dot{s}U \mid U \in \mathsf{Op}\, X, s \in PU\}.$$

That is, the open sets of $\Lambda_P$ are unions of sets in $\mathcal{B}_P$.

*Proof.* Since $\Lambda_P$ has the finest topology with respect to all $\dot{t}$ where $t \in PV$ for some $V \in \mathsf{Op}\, X$, a subset $W \subseteq \Lambda_P$ is open if and only if $\dot{t}^{-1}W \subseteq V$ is open for all $\dot{t}$, which is true if and only if $\dot{t}^{-1}W \in \mathsf{Op}\, X$. We wish to show that this occurs if and only if $W$ is a union of $\dot{s}U$'s.
($\Longrightarrow$) Observe that any $[U, s]_x = \dot{s}x \in W$ is contained in $\dot{s}(\dot{s}^{-1}W) \subseteq W$, which is in our desired form since $\dot{s}^{-1}W$ is open by hypothesis.
($\Longleftarrow$) Since preimage preserves unions, it suffices to show that $\dot{t}^{-1}\dot{s}U \subseteq V$ is open for all $U, V \in \mathsf{Op}\, X$, all $s \in PU$ and all $t \in PV$. Observe that $x \in \dot{t}^{-1}\dot{s}U$ if and only if $\dot{t}x \in \dot{s}U$, but since $\dot{t}x \in P_x$ we then have $\dot{t}x = \dot{s}x$, or $[V, t]_x = [U, s]_x$. We then have an open neighborhood $x \in W \subseteq U \cap V$ on which $t$ and $s$ agree on $W$, which then means $\dot{t}$ and $\dot{s}$ agree, so $W \subseteq \dot{t}^{-1}\dot{s}U$ as desired. □

Now we show that $\Lambda_P$ does indeed have the desired universal property.

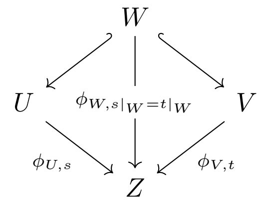
**Proposition 3.9.** The construction $\Lambda_P$ satisfies universal property (3.4), that is

$$\Lambda_P \cong \int^{U \in \mathsf{Op}\, X} PU \cdot U \in \mathsf{Top}\,.$$

*Proof.* We have an obvious candidate for the morphisms defining the universal cowedge into $\Lambda_P$, the maps $\dot{s} \colon U \to \Lambda_P$, which clearly satisfy (3.6). We want to show that any cowedge defined by $\{\phi_{U,s} \colon U \to Z\}_{U \in \mathsf{Op}\, X, s \in PU}$ factors uniquely through these maps:

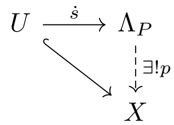
$$\begin{array}{ccc} U & \xrightarrow{\forall \dot{s}} & \Lambda_P \\ & {\scriptstyle \forall \phi_{U,s}} \searrow & \Big\downarrow {\scriptstyle \exists ! \phi} \\ & & Z \end{array}$$

By the commutativity of the diagram, $\phi$ necessarily sends any $[U, s]_x = \dot{s}x \in \Lambda_P$ to $\phi_{U,s}(x)$, so we have uniqueness. To show existence, we need to show that $\phi$ is well defined and continuous. By the universal property of the finest topology, $\phi$ is continuous if and only if each composite $\phi \circ \dot{s} = \phi_{U,s}$ is continuous, which we have by our hypothesis. Suppose $[U, s]_x = [V, t]_x \in \Lambda_P$ are two representatives for a germ. Then, there exists some $x \in W \subseteq U \cap V$ on which $s$ and $t$ agree, and hence the diagram



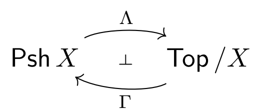

commutes using (3.6), and $\phi([U,s]_x) = \phi([V,t]_x)$ is well-defined. ◻

Using Proposition 3.3, we see that the coend bundle map $p \colon \Lambda_P \to X$ is given by the factorization




which sends $[U,s]_x = \dot{s}x$ to $x$. In other words, $p$ is the projection map that sends each stalk $P_x$ to $x$.

To summarize, we have the following desription for the adjunction in (3.1): (note that we are abusing notation to write $\Lambda$ for the bundle as well as the total space)

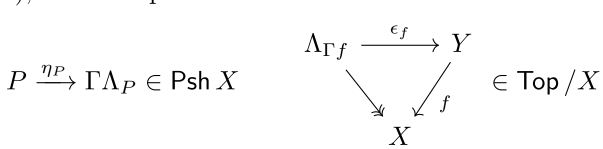
 (3.10)


In the next section, we will show that this adjunction restricts to an equivalence between the full subcategories of sheaves and étale spaces.

## 4. Sheaves and Étale Spaces

An immediate consequence of Proposition 3.8 is that each $\dot{s}$ is open. In addition, the bundle map $p \colon \Lambda_P \to X$ is a *local homeomorphism*, or *étale* in the sense that for all $e \in \Lambda_P$, there exists an open neighborhood $e \in V \subseteq \Lambda_P$ such that $pV \subseteq X$ is open and the restriction $p|_V \colon V \xrightarrow{\cong} pV$ is a homeomorphism. To see this, let $e = [U,s]_x$ be a choice of a representative for the equivalence class, and observe that $[U,s]_x \in \dot{s}U$ is an open neighborhood that is mapped homeomorphically to $U \subseteq X$ under $p$ with the inverse being $\dot{s}$.

**Proposition 4.1.** Let $\eta$ and $\epsilon$ denote respectively the unit and counit for the adjunction (3.10), with components




for presheaves $P$ and bundles $f \colon Y \to X$. Then, $\eta_P$ is an isomorphism if and only if $P$ is a sheaf, and $\epsilon_f$ is an isomorphism if and only if $f$ is étale.

*Proof.* First, we note that the essential image of $\Gamma$ lands in sheaves since we can glue continuous functions between topological spaces: let $U = \bigcup U_i$ be an open cover, and let $s_i \in \Gamma_f(U_i)$ be sections $s_i \colon U_i \to Y$ that agree on intersections $U_i \cap U_j$. The glued section $s\colon X \to Y$ by $s(x) := s_i(x)$ is then well-defined and continuous, since the preimage of any open $W \subseteq Y$ is $s^{-1}(W) = \bigcup s_i^{-1}(W)$. Hence, $\Gamma\Lambda_P$ is a sheaf, and so is $P$ if $\eta_P$ is an isomorphism.

Now suppose that $P$ is a sheaf. We wish to show that $\eta_P$ is a natural isomorphism, which means that for each $U \in \mathsf{Op}\, X$, the component $\eta_{PU}\colon PU \to \Gamma\Lambda_P U \in \mathsf{Set}$ is bijective. One can check using the proof of Proposition 2.5 that the unit map sends an (abstract) section $s \in PU$ to the induced (actual) section $\dot{s}\colon U \to \Lambda_P \in \Gamma\Lambda_P U$. To show injectivity, let $s, t \in PU$ such that $\dot{s} = \dot{t}$. Then, for all $x \in U$, we have

$$\dot{s}x = \dot{t}x \iff [U, s]_x = [U, t]_x \iff \exists x \in V_x \subseteq U \text{ open such that } s|_{V_x} = t|_{V_x}\,,$$

and by separability on the cover $U = \bigcup V_x$ we have $s = t$.

To show surjectivity, let $h \in \Gamma\Lambda_P U$ be a section $h\colon U \to \Lambda_P U$. For every $x \in U$, choose a representative $hx = [U_x, s_x]_x$. By Lemma 3.8, each $\dot{s_x} U_x$ is open in $\Lambda_P$, and by the continuity of $h$ there exists open neighborhoods $x \in V_x \subseteq U_x$ for each $x \in U$ such that $hV_x \subseteq \dot{s_x} U_x$, which means that for all $y \in V_x$, $hy$ is in both $P_y$ and $\dot{s}_x U_x$. Since $\dot{s}_x$ maps each $z \in U_x$ to something in $P_z$, we have

$$P_y \cap \dot{s_x} U_x = \{\dot{s_x} y\} \implies hy = \dot{s}_x(y).$$

Hence, for all $x, y \in U$, $\dot{s_x}$ and $\dot{s_y}$ agree with $h$ and therefore with each other on the intersection $V_x \cap V_y$. We again can then on the cover $U = \bigcup V_x$ to obtain an abstract section $s \in PU$ that restricts to $s_x$ on $V_x$, whose induced map $\dot{s}$ is equal to $h$ since the restriction of the continuous maps $\dot{s}|_{V_x} = \dot{s_x}|_{V_x}$ (equality by naturality) is locally equal to $h$, allowing us to apply gluing on $\Gamma\Lambda_P$.

Finally, we would like to show that if the bundle $f\colon Y \to X$ is étale, then $\epsilon_f$ is an isomorphism (the reverse direction is given by the fact that $\Lambda_P$ is étale over $X$). Again, check using Proposition 2.5 that $\epsilon_f$ sends a germ $[U, s]_x \in \Gamma_{\Lambda_P}$ to $sx \in Y$, where open sets $U \subseteq X$ and sections $s\colon U \to Y$. We define an inverse to $\epsilon_f$ by considering the inverses to the local homeomorphisms. For all $y \in Y$, let $y \in V_y \subseteq Y$ be an open neighborhood that is mapped homeomorphically onto $fV_y$ with inverse $s_y$, and define $\epsilon_f^{-1}(y) := \dot{s}_y f(y)$, noting that the definition is independent of the choice of $V_y$ and $s_y$ (two choices will agree on their intersection since they both define inverses to $f$). For the same reason, $s_y$ and $s_z$ agree on $f(V_y \cap V_z)$ for any $y, z \in Y$. We verify that $\epsilon_f^{-1}$ is continuous: for any $U \subseteq X$ open and any section $s\colon U \to Y$, we have $y \in \epsilon_f^{-1}(\dot{s}U)$ if and only if $\dot{s}_y f(y) = \dot{s} f(y)$. Take the open neighborhood $f(y) \in W \subseteq fV_y \cap U$ on which the germs agree, and consider $y \in f^{-1}W \subseteq V_y$ which serves as an open neighborhood around $y$ in $\epsilon_f^{-1}$. ☐

**Proposition 4.2.** Let $\mathcal{C} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}$ be an adjunction with unit $\eta$ and counit $\epsilon$. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory of *fixed points* of the monad asscociated to the adjunction. Similarly, let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the full subcategory of fixed points of the comonad. Then, the adjunction restricts to an adjoint equivalence between $\mathcal{C}_0$ and $\mathcal{D}_0$.

Furthermore, if the objects in the essential image of $L$ and $R$ are fixed points, then the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is reflective with reflector (left adjoint) $RL$, and dually, $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ is coreflective with coreflector $LR$.

*Proof.* First, we verify that the images of fixed points are fixed points using the triangle identities: for any $c \in \mathcal{C}_0$, we note that $\epsilon_{Lc}$ defines an inverse to the isomorphism

$L\eta_c$, so it is necessarily also an isomorphism (dually, $\epsilon_{Rd}$ is an isomorphism for all $d \in \mathcal{D}_0$). Then, observe that by definition, the unit and counit being isomorphisms on fixed points means that the adjunction restricts to a adjoint equivalence $\mathcal{C}_0 \simeq \mathcal{D}_0$. Finally, the reflectivity statement is given by the following composition of natural isomorphisms: for all $c_0 \in \mathcal{C}_0$ and all $c \in \mathcal{C}$,

$$\begin{aligned} \mathcal{C}(c, c_0) &\cong \mathcal{C}(c, RLc_0) && \text{(fixed point)} \\ &\cong \mathcal{D}(Lc, Lc_0) && \text{(adjunction)} \\ &\cong \mathcal{C}(RLc, RLc_0) && \text{(equivalence)} \\ &\cong \mathcal{C}(RLc, c_0) && \text{(fixed point)} \end{aligned}$$

and the dual statement follows analogously. □

Finally, combine Propositions 4.1 and 4.2 and we have the following results:

**Corollary 4.3.** The adjunction (3.10) restricts to an adjoint equivalence between the full subcategories of sheaves and étale spaces.

**Corollary 4.4.** The category of sheaves on $X$ is a reflective subcategory of presheaves on $X$, with the reflector (called *sheafification*) given by $\Gamma\Lambda$.

*Remark* 4.5. In particular, Corollary 4.4 tells us how to construct limits and colimits in the category of sheaves. First, limits and colimits in presheaves (or any other functor category) are constructed objectwise (see [Rie16], 3.3.9). Limits in a reflective subcategory are created by the inclusion map, and colimits are formed by applying the reflector to the colimit constructed in the parent category (see [Rie16], 4.5.15). In our case, limits in sheaves are defined taking limits on each open set; colimits are defined by first taking colimits on each open set (which doesn't always form a sheaf), then sheafifying. This phenomenon often causes limits to be more well behaved than colimits when dealing with sheaves — for instance, the global sections functor is left but not usually right exact, prompting the study of sheaf cohomology.

## 5. Useful Consequences

**Corollary 5.1.** The following are equivalent ways to describe a morphism of sheaves over $X$:

(a) A natural transformation $h\colon F \to G$;
(b) A map of bundles $\Lambda_h\colon \Lambda_F \to \Lambda_G$ over $X$;
(c) A family $h_x\colon F_x \to G_x$ of functions between stalks at each $x \in X$ such that for all $U \in \mathsf{Op}\, X$ and all $s \in FU$, the function $x \mapsto h_x \dot{s} x$ from $U$ to $\Lambda_G$ is continuous.

*Proof.* The correspondance between (a) and (b) is given by the equivalence in Corollary 4.3. In particular, the image of a natural transformation $h\colon F \to G$ under $\Lambda$ is given by the canonical map between coends

$$\begin{array}{ccccc} \int^{\mathsf{Op}\,X} FU \cdot U & \longleftarrow & FU \cdot U & \xleftarrow{\iota_s} & U \\ \big\downarrow{\scriptstyle \exists !} & & \big\downarrow{\scriptstyle h_u \cdot U} & & \big\| \\ \int^{\mathsf{Op}\,X} GU \cdot U & \longleftarrow & GU \cdot U & \xleftarrow{\iota_{h_U s}} & U \end{array} \iff \begin{array}{ccc} \Lambda_F & \xleftarrow{\dot{s}} & U \\ \big\downarrow{\scriptstyle \exists ! \Lambda_h} & \swarrow{\scriptstyle \dot{h_u s}} & \\ \Lambda_G & & \end{array}$$

which sends a germ $[U, s]_x \in \Lambda_F$ to $[U, h_U s]_x \in \Lambda_G$.

The equivalence between (b) and (c) is obtained through the universal property of $\Lambda_F$. A function of sets $\left(\Lambda_F = \coprod_{x\in X} F_x\right) \to \Lambda_G$ is equivalent to a set of functions $F_x \to \Lambda_G$, and the preservation of bundle maps over $X$ translates to the fact that the component maps restrict to $F_x \to G_x$. The continuity condition is again given by the universal property of the strong topology. □

This corollary illustrates the local nature of sheaves: morphisms of sheaves are determined by the induced mapping on stalks, which we can think of as "infinitesimal" data around each point. This viewpoint is emphasized in the following convenient result.

**Proposition 5.2.** A map of sheaves is a monomorphism (resp. epimorphism) if and only if it induces an injection (resp. surjection) on all stalks.

*Proof.* See [MM94], Proposition II.6.6. □

## References

[MM94] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic*. New York, NY: Springer, 1994.

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