PLAN

1) Categories
2) monoids
3) integers
4) monoidal Cat's.

CATEGORIES

IDEA math things & things that go btwn them

EX. 1) Sets + functions

2) vector spaces + linear maps

3) groups + group morphisms

4) top. spaces + cts fcn's

5) CAT: cat's + functors

$F: \mathcal{C} \to \mathcal{D}$ st

$$\begin{array}{ccc} X & \mapsto & FX \\ f\downarrow & \mapsto & \downarrow Ff \\ Y & \mapsto & FY \end{array}$$

"maps to"

$Fg \cdot Ff = F(g \cdot f)$

$F(1_X) = 1_{FX}$

Defn: A category $\mathcal{C}$ consists of

→ objects $X, Y \in \mathrm{ob}\,\mathcal{C}$

→ morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\ni f: X \to Y$

w/

→ identity $1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$

→ $\forall f: X \to Y$, $\forall g: Y \to Z$, composite $gf: X \to Z$

st

→ (associative) $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$

$h(gf) = (hg)f$

→ (unital) $f: X \to Y$

$f 1_X = f = 1_Y f$

We call $f: X \to Y$ an isomorphism if $\exists g: Y \to X$ st $gf = 1_X$, $fg = 1_Y$

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MONOIDS

IDEA groups, but no inverses.

Defn: A monoid is a set S w/ a fcn $S \times S \xrightarrow{\cdot} S$ st $(a,b) \mapsto a \cdot b$

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a,b,c \in S$
- $\exists e \in S$ st $\forall a \in S$ $a \cdot e = e \cdot a = a$

EX. 1) $(\mathbb{N} \cup \{0\}, +, 0)$

2) $(\mathbb{N}, \times, 1)$

3) $n \times n$ matrices, mult., I

EX. 4) $\{T, F\} = S$

$\cdot$ is AND ($\wedge$). $e = T$.

$T \wedge T = T$. $T \wedge F = F$

$F \wedge F = F$.

5) $\{T, F\}$ OR ($\vee$). $e = F$.

$F \vee F = F$, $T \vee F = T$.

$T \vee T = T$.

Rmk neither 4) or 5) are $\mathbb{Z}/2\mathbb{Z} \Rightarrow$ not groups

6) subsets, union/intersectn, $\emptyset$ / whole set

7) groups $\forall a \in S$ $\exists b \in S$ st $ab = ba = e$.

②

T.

e=F.

T.

5)

groups

intersectn

whole set

S

a = e

③

GOAL Describe monoids categorically.

$S\times S \xrightarrow{\mu} S$, $\{*\} \xrightarrow{e} S$

$$\begin{tikzcd}
(a,b,c) \arrow[r, mapsto] \arrow[d, mapsto] & (a\cdot b,c) \arrow[d, mapsto] \\
(a,b\cdot c) \arrow[r, mapsto] & (a\cdot b)\cdot c = a\cdot(b\cdot c)
\end{tikzcd}
\qquad
\begin{tikzcd}
S\times S\times S \arrow[r, "\mu\times id"] \arrow[d, "id\times\mu"'] & S\times S \arrow[d, "\mu"] \\
S\times S \arrow[r, "\mu"'] & S
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, mapsto] & (a,e) \arrow[d, mapsto] \\
a & a\cdot e
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "id\times e"] \arrow[dr, "id"'] & S\times S \arrow[d, "\mu"] \\
& S
\end{tikzcd}$$

$a = a\cdot e$ (and $\triangleright$)

(groups - $\eta: S \to S$ + diagram)

Why?

- $\triangleright$ top. groups/monoids $\in$ Top.
- $\triangleright$ Lie groups $\in$ Man.
- $\triangleright$ Abelian groups/monoids $\in$ Grp/Monoid (Eckmann-Hilton).

4

$Ab \ni (\mathbb{Z}, +, 0)$. $(\mathbb{Z}, \times, 1)$

Q Can we get our usual $\mathbb{Z}$ w/ such a construction?

ATTEPT 1 Interpret monoid axioms of $(\mathbb{Z}, \times, 1)$ in Ab.

$\mathbb{Z} \times \mathbb{Z} \xrightarrow{\mu} \mathbb{Z}$ group mor.

$$a \times (b+c) = \mu(a+0, b+c)$$

group product defn

$$= \mu((a,b)+(0,c))$$

$$= \mu(a,b)+\mu(0,c)$$

$$= a\times b + 0 \times c = a \times b$$

want $a\times(b+c) = a\times b + a \times c$

$(a+b)\times c = a \times c + b \times c$

$\to$ "bilinear" map

TENSOR PRODUCT

$A, B, C \in Ab$.

$$\left\{ \begin{matrix} A \times B \to C \\ \text{bilinear} \end{matrix} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{matrix} A \otimes B \to C \\ \text{group mor} \end{matrix} \right\}$$

Defn: $a \otimes b = [(a,b)]$

$$A \otimes B = F^{ab}(A \times B)/\sim$$

$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

5

ATTEPT 2 (still in Ab.)

$\mathbb{Z}\otimes\mathbb{Z} \xrightarrow{\mu} \mathbb{Z}$

$a\otimes b \longmapsto a\times b$

$a\times(b+c) = \mu(a\otimes(b+c))$

by our defn $\to$ $= \mu(a\otimes b + a\otimes c)$

group mor $\to$ $= \mu(a\otimes b)+\mu(a\otimes c)$

$= a\times b + a\times c$ ✓

Defn A monoidal cat. consists of

- A category M
- A functor $\otimes: M\times M \to M$

$$(x,y) \longmapsto x\otimes y$$
$$(f,g)\downarrow \longmapsto \downarrow f\otimes g$$
$$(x',y') \longmapsto x'\otimes y'$$

- identity obj $1 \in M$

st $\forall x,y,z\in M$ $(x\otimes y)\otimes z \overset{\alpha_{x,y,z}}{\cong} x\otimes(y\otimes z)$

$1\otimes x \overset{\lambda_x}{\cong} x \overset{\rho_x}{\cong} x\otimes 1$

isomorphisms

Ex. $(\mathrm{Set}, \times, \{*\})$

X set

$$X \times \{*\} = \{(x,*) \mid x \in X\}$$

$$\neq \qquad\qquad \neq$$

$$X \qquad\qquad X$$

Sim.

$$\begin{array}{ccc} (X \times Y) \times Z & \neq & X \times (Y \times Z) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ ((x,y),z) & & (x,(y,z)) \end{array}$$

BUT $X \times \{*\} \cong X$

$$(X \times Y) \times Z \cong X \times (Y \times Z)$$

bijections

X 3+ objects?

have iso's, but need to be same iso's.

Defn (Con't)

⬠ and $\triangle$ identities hold.

Thm (coherence, Mac Lane, 1964?)

"All diagrams commute"

6.5

EX.

1) $(\mathrm{Set}, \times, \{*\})$

2) $(\mathrm{Set}, \sqcup, \emptyset)$

$\Rightarrow$ product + terminal obj.
coproduct + initial obj.

3) $(\mathrm{Ab}, \otimes, \mathbb{Z})$

Claim: $A, B, C \in \mathrm{Ab}$

$A \otimes \mathbb{Z} \cong A$, $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

$\Rightarrow (R\text{-Mod}, \otimes, R)$

4) Top_*, smash product, pointed S^0