

PLAN

- 1) Categories
- 2) monoids
- 3) integers
- 4) monoidal
Cat's.

CATEGORIES

IDEA math things
 & things that go b/w them

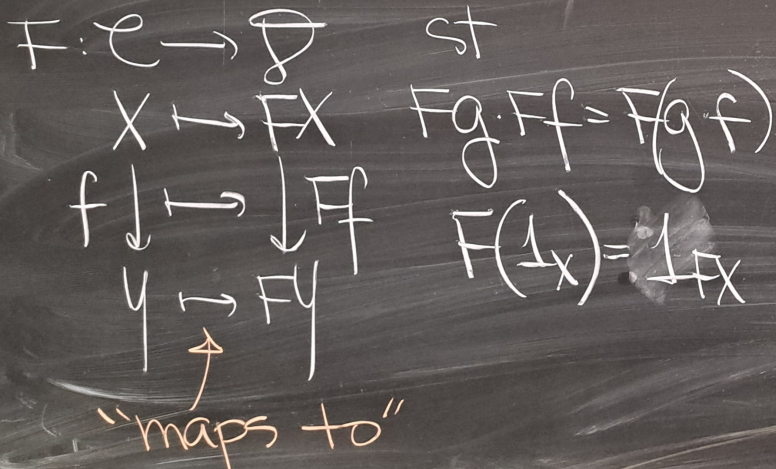
EX. 1) Sets + functions

2) vector spaces
 + linear maps

3) groups + group
 morphisms

4) top. spaces + cts fcn's

5) CAT: cat's + functors



1

Defn A category \mathcal{C} consists of

→ objects $x, y \in \text{ob } \mathcal{C}$

→ morphisms $\text{Hom}_{\mathcal{C}}(x, y)$

→ identity $1_x \in \text{Hom}_{\mathcal{C}}(x, x)$

→ $\forall f: x \rightarrow y, g: y \rightarrow z$,
 composite $gf: x \rightarrow z$

st (associative) $h \circ (g \circ f) = (h \circ g) \circ f$

→ (unital) $f \circ 1_x = f = 1_y \circ f$

We call $f: x \rightarrow y$ an
isomorphism if $\exists g: y \rightarrow x$

st $gf = 1_x, fg = 1_y$

MONOIDS

of IDEA groups, but
no inverses.

Defn: A monoid is
a set S w/ a fcn
 $S \times S \rightarrow S$ st
 $(a, b) \mapsto a \cdot b$

$$\rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$\forall a, b, c \in S$

$$\rightarrow \exists e \in S \text{ st } \forall a \in S$$

$$a \cdot e = e \cdot a = a$$

EX. 1) $(\mathbb{N} \cup \{0\}, +, 0)$

2) $(\mathbb{N}, \times, 1)$

3) $n \times n$ matrices,
mult., I

(2)

EX. 4) $\{T, F\} = S$

- \bullet - is AND (\wedge) $e = T$

$$T \wedge T = T \quad T \wedge F = F$$

$$F \wedge F = F$$

5) $\{T, F\}$ OR (\vee) $e = F$

$$F \vee F = F, \quad T \vee F = T$$

$$T \vee T = T$$

Rmk neither 4) or 5)

are $\mathbb{Z}/\mathbb{Z} \Rightarrow$ not groups

6) subsets, union/intersectn
 \emptyset / whole set

7) groups $\forall a \in S$

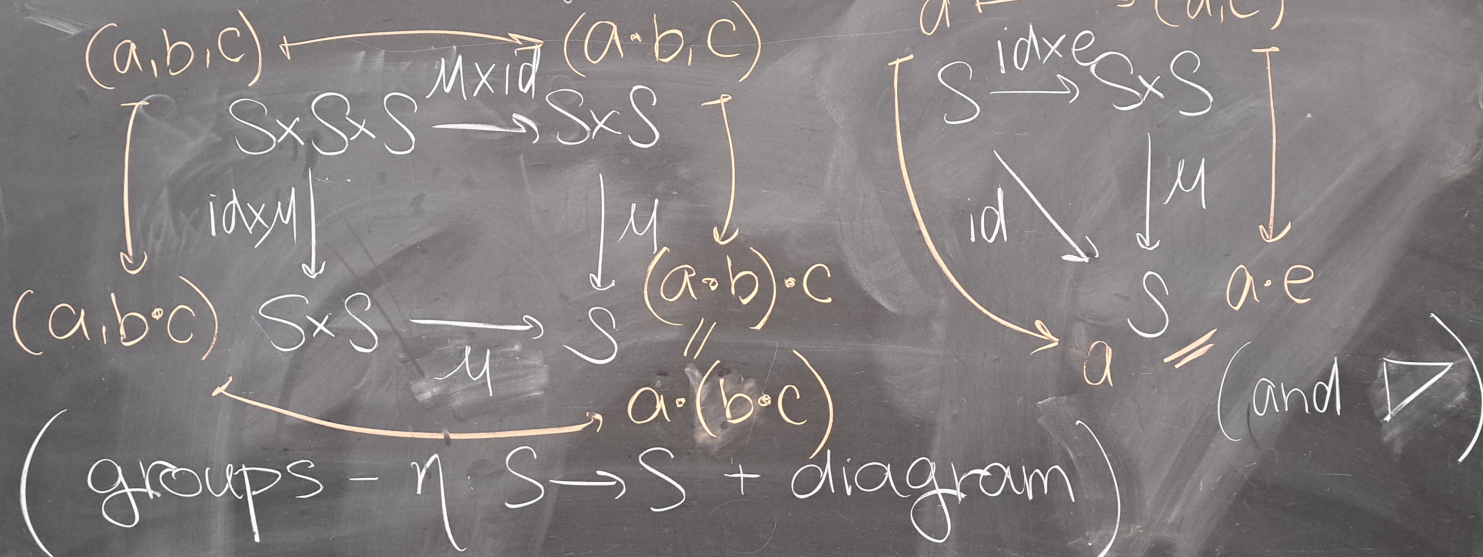
$$\exists b \in S \text{ st } ab = ba = e$$

(2)

(3)

GOAL Describe monoids categorically.

$$S \times S \xrightarrow{\mu} S, \{*\} \xrightarrow{e} S$$



- Why?
- \rightarrow top. groups/monoids \in Top.
 - \rightarrow Lie groups \in Man.
 - \rightarrow Abelian groups/monoids \in Grp/Monod.
- (Eckmann-Hilton)

T

$e = F$

T

5)

groups

intersectn

hole set

S

$a = e$

$$Ab \Rightarrow (\mathbb{Z}, +, 0), (\mathbb{Z}, \times, 1)$$

Q Can we get our usual \mathbb{Z} w/ such a construction?

ATTEMPT 1 Interpret monoid axioms of $(\mathbb{Z}, \times, 1)$ in Ab .

$\mathbb{Z} \times \mathbb{Z} \xrightarrow{M} \mathbb{Z}$ group mor.

$$ax(b+c) = M(a+0, b+c)$$

group product defn \rightarrow $M((a,b) + (0,c))$

$$= M(a,b) + M(0,c)$$

$$= axb + 0xc = axb$$

(4)

want $ax(b+c) = axb + axc$
 $(a+b)xc = axc + bxc$

\rightarrow "bilinear" map

TENSOR PRODUCT

$$A, B, C \in Ab.$$

$$\left\{ \begin{array}{l} A \times B \rightarrow C \\ \text{bilinear} \end{array} \right\} \xleftrightarrow{bij} \left\{ \begin{array}{l} A \otimes B \rightarrow C \\ \text{group mor} \end{array} \right\}$$

Defn: $a \otimes b = [(a,b)]$

$$A \otimes B = F^{Ab}(A \times B) / \sim$$

$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

(5)

ATTEPT 2 (still in Ab.)

$$\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{M} \mathbb{Z}$$

$$a \otimes b \mapsto a \cdot b$$

$$a \cdot (b+c) = M(a \otimes (b+c))$$

by our defn $\rightarrow = M(a \otimes b + a \otimes c)$

group mor $\rightarrow = M(a \otimes b) + M(a \otimes c)$
 $= a \cdot b + a \cdot c$



Defn A monoidal cat. consists of

\rightarrow A category M

\rightarrow A functor $\otimes : M \times M \rightarrow M$
 $(x, y) \mapsto x \otimes y$

$(f, g) \downarrow \mapsto \downarrow f \otimes g$
 $(x', y') \mapsto x' \otimes y'$

\rightarrow identity obj $1 \in M$

st $\forall x, y, z \in M$
 $(x \otimes y) \otimes z \stackrel{\alpha_{x,y,z}}{=} x \otimes (y \otimes z)$

$1 \otimes x \stackrel{\beta_x}{=} x \stackrel{\gamma_x}{=} x \otimes 1$

isomorphisms

Ex. (Set, $X, \{*\}$)

X set

$$X \times \{*\} = \{(x, *) \mid x \in X\}$$

~~#~~
~~X~~

~~∩~~
~~X~~

Sim. $(X \times Y) \times Z \neq X \times (Y \times Z)$
 $((x, y), z) \quad (x, (y, z))$

BUT $X \times \{*\} \cong X$
 $(X \times Y) \times Z \cong X \times (Y \times Z)$
↑
↑
bijections

⑥

⊗ 3+ objects?

have iso's, but
need to be same
iso's.

Defn (con't).

□ and △ identities
hold.

Thm (Coherence,
Mac Lane, 1964?)

"All diagrams commute"

6.5

EX.

1) $(\text{Set}, X, \{*\})$

2) $(\text{Set}, \sqcup, \emptyset)$

\rightarrow product + terminal obj.
coproduct + initial obj.

3) $(\text{Ab}, \otimes, \mathbb{Z})$

Claim: $A, B, C \in \text{Ab}$.

$$A \otimes \mathbb{Z} \cong A, (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

\rightarrow $(R\text{-Mod}, \otimes, R)$

4) Top_* , Smash product,
pointed S^0