Recall: derivatives on $\mathbb{R}$: $f \in C^\infty(\mathbb{R})$

$$x \in \mathbb{R} \Rightarrow f'(x) := \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

Q. Generalize to $\mathbb{R}^n$? $f \in C^\infty(\mathbb{R}^n)$ "tangent space at x"

A. Directional derivatives: $x \in \mathbb{R}^n$ $v \in T_x\mathbb{R}^n \cong \mathbb{R}^n$

$$(d_v f)_x := \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon v) - f(x)}{\varepsilon} \in \mathbb{R}$$

Think of tangent vectors as "directions to take derivative in"; in multivariable calc. we pick a basis for $T_x\mathbb{R}^n$ and consider partial derivatives, but it's more natural just to consider all directions when defining derivative of f:

$$(df)_x : T_x\mathbb{R}^n \to \mathbb{R} \quad \text{linear map}$$
$$v \mapsto (d_v f)_x$$

i.e. $(df)_x \in \mathcal{L}(T_x\mathbb{R}^n, \mathbb{R}) =: T_x^*\mathbb{R}^n$

"cotangent space"

Lastly, at each $x \in \mathbb{R}^n$ associate the derivative of f:

$$df : (x \in \mathbb{R}^n) \longrightarrow T_x^* \mathbb{R}^n \qquad \text{Codomain is "dependent" on } x$$
$$x \mapsto (df)_x$$

Def. (cotangent bundle)

$$T^*\mathbb{R}^n := \coprod_{x \in \mathbb{R}^n} T_x^* \mathbb{R}^n \quad + \text{ a topology}$$

$$\text{proj} \downarrow \qquad (x \in \mathbb{R}^n, \phi \in T_x^* \mathbb{R}^n)$$
$$\mathbb{R}^n \qquad\qquad \downarrow$$
$$x$$

$\Rightarrow$ $df : \mathbb{R}^n \longrightarrow T^*\mathbb{R}^n$ "section", i.e.

$$T^*\mathbb{R}^n$$
$$\circlearrowleft \downarrow \quad \text{identity, or } x \mapsto (x, v_x) \mapsto x$$
$$\mathbb{R}^n$$

Def. $M \in \text{SmMfld}$ locally $\mathbb{R}^n$ + properties

Differential 1-forms :=

{sm. sections of the cotangent bundle}

$\rightarrow$ Denote $\Omega^1(\mathbb{R}^n)$

Differential k-forms are obtained "formally". $\Omega^k(M)$.

<u>Ex.</u> Define $x : \mathbb{R}^2 \to \mathbb{R}$

$(a,b) \mapsto a$

$\Rightarrow \quad dx \in \Omega^1(\mathbb{R}^2)$

$dx \wedge dy \in \Omega^2(\mathbb{R}^2)$ "volume form"

Def/<u>Thm</u>. (de Rham complex)

$\exists! \, d$ $\mathbb{R}$-linear st

$$C^\infty(M) =: \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

st 1) $f \mapsto df$ (for $f \in \Omega^0(M)$)

2) $d^2 = 0$

3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p (\alpha \wedge d\beta)$

w/ $p = \deg \alpha$.

Ex. $C^\infty(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \cong C^\infty(\mathbb{R})$

$f \mapsto f'$

kernel: $df \equiv 0 \Rightarrow f$ constant

"closed"

$$C^\infty(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R})$$

$\int_0^x g \ \overset{\rightarrow}{\exists} \ g$ "exact"

$$0 \to \mathbb{R} \hookrightarrow C^\infty(\mathbb{R}) \twoheadrightarrow \Omega^1(\mathbb{R}) \to 0$$

"short exact sequence"

de Rham cohomology: studies the failure of the de Rham complex to be exact.

Thm. (de Rham) $M \in \mathrm{SmMfld}$

$$H^\bullet_{dR}(M) \cong H^\bullet(M, \mathbb{R})$$

$\omega \in \Omega^k(M)$ $[\omega] \mapsto \left([C] \mapsto \int_C \omega \right)$

$C : \Delta^k \to M$

closed singular k-chain