Braidings on Non-Split Tambara-Yamagami Categories over the Reals

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Preliminaries

Definition: A *fusion category* is a rigid finite semisimple linear monoidal category. (*Idea: categorified rings; replace "equal" with "isomorphic"*)

Ring R	Fusion Category $\mathcal{C}$
R set, elements $a, b, c \in R$	objects $A, B, C \in \mathrm{ob}(\mathcal{C})$ morphisms $A \xrightarrow{f} C$, $B \xrightarrow{g} D$
sums $a + b$	direct sums $A \oplus B$
products $a \times b$	tensor products $A \otimes B$ (bifunctor) $A \otimes B \xrightarrow{f \otimes g} C \otimes D$
associativity $(ab)c = a(bc)$	associators $(A \otimes B) \otimes C \xrightarrow[\cong]{\alpha_{A,B,C}} A \otimes (B \otimes C)$

The coherence of associators is given by the pentagon axiom.

Definition: Let $\mathcal{C}$ be a fusion category over $\mathbb{F}$ (hom-sets are $\mathbb{F}$ vector spaces). A simple object $X \in \mathcal{C}$ (has no non-trivial quotients) is *split* if $\mathrm{End}(X) \cong \mathbb{F}$. A category is split if all of its simple objects are.

Lemma (Schur): A morphism between simple objects is either zero or an isomorphism. The endomorphisms of a simple object is a division algebra over $\mathbb{F}$.

Definition: Let G be a finite group. The *Tambara-Yamagami fusion ring* $\mathsf{TY}(G)$ has a $\mathbb{Z}$-basis $G \sqcup \{m\}$, $m \notin G$. The product is defined as follows

$$a \cdot b = ab\,, \quad a \cdot m = m = m \cdot a\,, \quad m \cdot m = N \cdot \sum_{c \in G} c\,,$$

for all $a, b \in G$ with $N \in \mathbb{N}$. A *Tambara-Yamagami category* is a categorification of a TY ring (the elements of the basis are the simple objects of the category).

Associator Structure

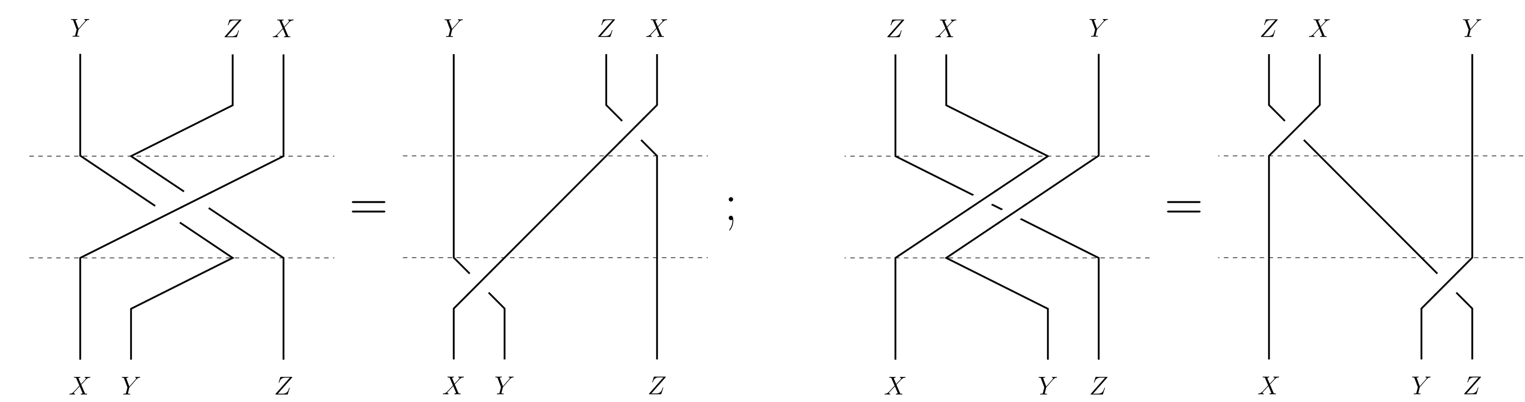
By the Yoneda lemma, we can completely determine the associators by looking at what happens when we precompose by them. This lets us explicitly compute the constraint given by the pentagon axiom using string diagrams and linear algebra after choosing bases for hom-spaces (denoted by trivalent vertices).

> **Theorem:** [Tambara-Yamagami '98] Any split TY category is determined by a triple (G, χ, τ), where G is a finite group, $\chi\colon G \times G \to \mathbb{R}^\times$ is a nondegenerate symmetric bicharacter, and $\tau \in \{\pm 1/\sqrt{|G|}\}$.

A breakdown of possible endomorphism combinations of simple objects over non-split TY categories as well as a complete classification of associators can be found in [PSS23]. The classification for the real-quaternionic case discussed in this poster differs from the split case only in the constant τ.

Braidings

Definition: A *braiding* on a monoidal category $\mathcal{C}$ is a (natural) set of isomorphisms $c_{X,Y}\colon X \otimes Y \xrightarrow{\cong} Y \otimes X$ for all objects $X, Y \in \mathcal{C}$ satisfying the following conditions (hexagon identities):



In a TY category, precomposing by braiding isomorphisms gives us functions $\sigma_{0,1,2,3}$ with inputs in G and outputs in the endomorphism algebras of simple objects, which completely determine the braiding structure.

Objective

Classify all possible braiding structures on split and non-split Tambara-Yamagami categories over $\mathbb{R}$ up to monoidal equivalence.

Results

Performing diagrammatic computations (see example) turns the hexagon identities into a set of sixteen equations, which we simplified to obtain

$$\sigma_0(a,b) = \chi(a,b), \tag{1}$$
$$\sigma_1(a)^2 = \chi(a,a), \tag{2}$$
$$\sigma_1(ab) = \sigma_1(a)\sigma_1(b)\chi(a,b), \tag{3}$$
$$\sigma_2(a) = \sigma_1(a), \tag{4}$$
$$\sigma_3(1)^2 = \tau \sum_{c \in G} \sigma_1(c), \tag{5}$$
$$\sigma_3(a) = \sigma_3(1)\sigma_1(a)\chi(a,a). \tag{6}$$

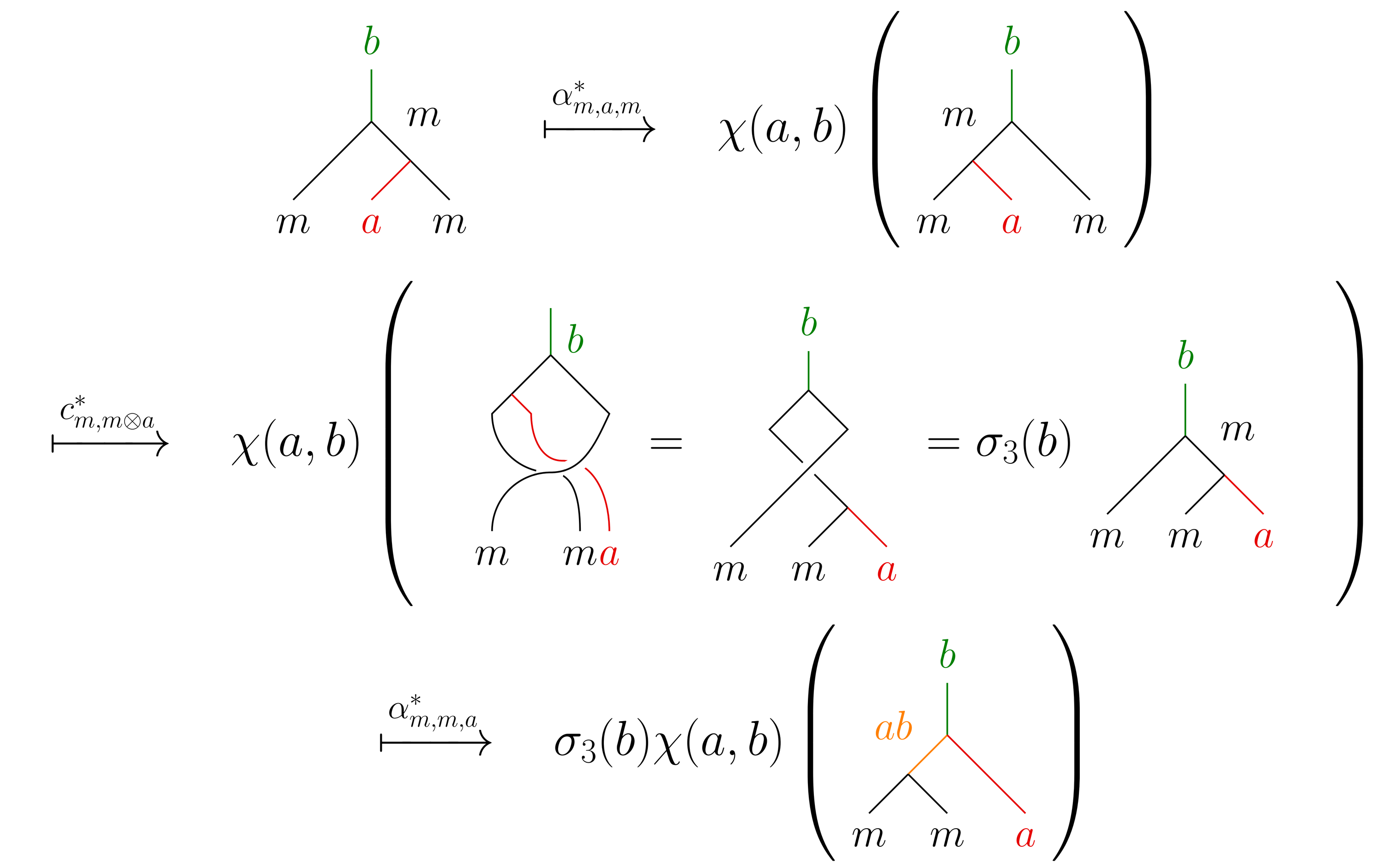
Equation (2) tells us that $\chi(a,a) > 0$ for all $a \in G$, which places a big restriction on χ (see [Wal63]), leading to the following classification.

> **Theorem:** Any split TY category over $\mathbb{R}$ that admits a braiding is equivalent to $\mathcal{C}(K_4^n, h^{\oplus n}, \tau)$, where K_4 denotes the Klein four-group, h denotes the hyperbolic pairing on K_4, $\tau \in \{\pm 1/2^n\}$, and $n \in \mathbb{N}$. There are exactly two non-equivalent braidings on such a category.

The classification for one of the non-split cases (with $\mathrm{End}(1) \cong \mathbb{R}$ and $\mathrm{End}(m) \cong \mathbb{H}$) can be reduced down to the split case classification with a few difference in constants, as a naturality argument (see proof) shows that braiding coefficients must lie in $Z(\mathbb{H}) = \mathbb{R}$. The resulting simplified braiding equations differ from (1) through (6) by a coefficient of -2 in equation (5).

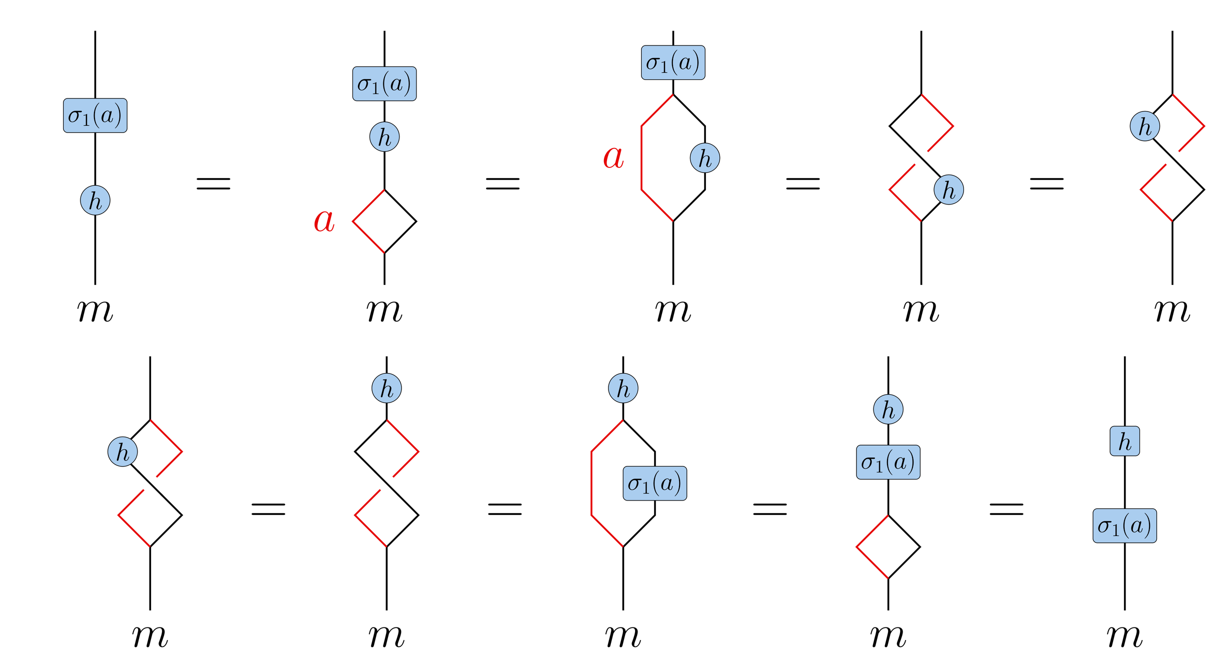
Example Computation

(Split case, hexagon 6, path 1)



Proof of Lemma for $\mathbb{R}/\mathbb{H}$ Case

We show that $\sigma_1(a)h = h\sigma_1(a)$ for any $a \in G$ and any $h \in \mathbb{H}$.



Acknowledgements

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