Braidings on Non-Split Tambara-Yamagami Categories over the Reals

Yoyo Jiang

Joint with David Green and Sean Sanford

March 9, 2024

Topics

Introduction

- Categories
- Fusion Categories / Tambara-Yamagami Categories
- Split Case (Example Computation)
- Non-Split Case: Real-Quaternionic

Categories

Idea: Mathematical objects and maps between them

- Sets and functions
- Vector spaces and linear transformations
- Groups and group homomorphisms
- Rings and ring homomorphisms
- Topological spaces and continuous functions

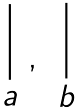
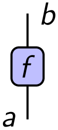
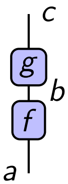
Definition

Categories

objects	$a, b, c \in \mathcal{C}$
morphisms	$(a \xrightarrow{f} b) \in \mathrm{Hom}(a, b)$
identity	$\exists(a \xrightarrow{\mathrm{id}_a} a) \in \mathrm{Hom}(a, a)$
composition	$\forall(a \xrightarrow{f} b)\ \forall(b \xrightarrow{g} c)\ \exists(a \xrightarrow{g \circ f} c)$
associativity	$\forall(a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d) \quad (hg)f = h(gf)$
unitality	$\forall(a \xrightarrow{f} b) \implies f\,\mathrm{id}_a = \mathrm{id}_b\,f = f$

String Diagrams

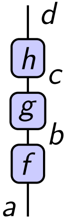
Categories

Objects: strings 	Morphisms: coupons 
Identity: empty string 	Composition: vertical stacking 

String Diagrams (Continued)

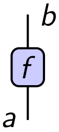
Categories

Associativity: correctness of



(no parenthesis)

Unitality: correctness of



(can compose with empty string)

Fusion Categories

Idea: “categorified” rings

Ring R	Fusion Category $\mathcal{C}$
R set, elements $a, b, c \in R$	objects $A, B, C \in \mathrm{ob}(\mathcal{C})$ morphisms $A \xrightarrow{f} C$, $B \xrightarrow{g} D$
sums $a + b$	direct sums $A \oplus B$
products $a \times b$	tensor products $A \otimes B$, $C \otimes D$ (functor) $A \otimes B \xrightarrow{f \otimes g} C \otimes D$
associativity $(ab)c = a(bc)$	<u>associators</u> $(A \otimes B) \otimes C \xrightarrow[\cong]{\alpha_{A,B,C}} A \otimes (B \otimes C)$

(Defn: A fusion category is a rigid finite semisimple linear monoidal category.)

Pentagon Axiom

Fusion Categories

Q. How to tensor four objects?
(In rings, $((ab)c)d = (a(bc))d = a((bc)d) = a(b(cd)) = (ab)(cd) =: abcd$)

In fusion categories, two ways to go between $((AB)C)D$ to $A(B(CD))$ (omitting $\otimes$s):

(continued on next slide)

Pentagon Axiom

Fusion Categories

Q. How to tensor four objects?

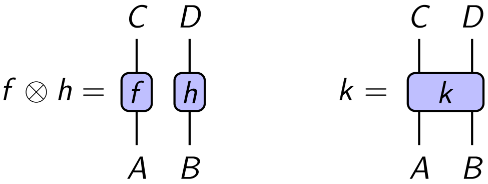
$$\begin{tikzcd}
((AB)C)D \arrow[r, "\alpha_{AB,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes \mathrm{id}_D"'] & (AB)(CD) \arrow[dr, "\alpha_{A,B,CD}"] & \\
(A(BC))D \arrow[dr, "\alpha_{A,BC,D}"'] & & A(B(CD)) \\
& A((BC)D) \arrow[ur, "\mathrm{id}_A \otimes \alpha_{B,C,D}"'] &
\end{tikzcd}$$

We require them to be the same (diagram commutes).

String Diagrams (Con't)

Fusion Categories

We denote tensor products on string diagrams by stacking boxes horizontally: $\forall(A \xrightarrow{f} C)$, $\forall(B \xrightarrow{h} D)$, $\forall(A \otimes B \xrightarrow{k} C \otimes D)$, we write



Braidings

Fusion Categories

Idea: Categorifying commutativity

Definition: A *braiding* on a fusion category $\mathcal{C}$ is a (natural) set of isomorphisms $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ for all objects $X, Y \in \mathcal{C}$, denoted as

satisfying the following conditions: (continued on next slide)

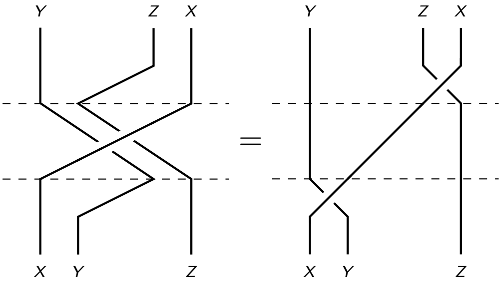
Braidings

Fusion Categories

Definition: (continued) 1) hexagon axiom

$$\begin{tikzcd}
& X(YZ) \arrow[r, "c_{X,YZ}"] & (YZ)X \arrow[dr, "\alpha"] & \\
(XY)Z \arrow[ur, "\alpha"] \arrow[dr, "c_{X,Y}\otimes \mathrm{id}_Z"'] & & & Y(ZX) \\
& (YX)Z \arrow[r, "\alpha"'] & Y(XZ) \arrow[ur, "\mathrm{id}_Y \otimes c_{X,Z}"'] &
\end{tikzcd}$$

$\iff$



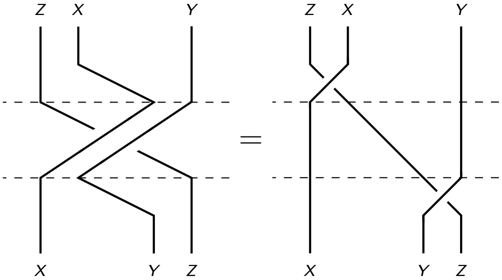
Braidings

Fusion Categories

and 2) inverse hexagon axiom

$$\begin{tikzcd}
& (XY)Z \arrow[r, "c_{XY,Z}"] & Z(XY) \arrow[dr, "\alpha^{-1}"] & \\
X(YZ) \arrow[ur, "\alpha^{-1}"] \arrow[dr, "\mathrm{id}_X \otimes c_{Y,Z}"'] & & & (ZX)Y \\
& X(ZY) \arrow[r, "\alpha^{-1}"'] & (XZ)Y \arrow[ur, "c_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

$\Longleftrightarrow$



Tambara-Yamagami Categories

Fusion Categories

Idea: Categorifying “near-group” rings

Let G be a finite group. The *(split) Tambara-Yamagami fusion category* has $G \sqcup \{m\}$ as a set of simple objects (“basis”), and objects are generated under finite direct sums. The tensor product is defined on simples: (for all $a, b \in G$)

$$a \otimes b = ab \;\;,\;\; a \otimes m = m = m \otimes a \;\;,\;\; m \otimes m = \bigoplus_{c \in G} c,$$

as $\otimes$ distributes over $\oplus$ (these are called “fusion rules”).

Split vs. Non-Split

Fusion Categories

(Helpful fact: fusion categories are linear, i.e. Hom's are vector spaces over $\mathbb{F}$.)

Definition: A simple object $X \in \mathcal{C}$ is *split* if $\mathrm{End}(X) := \mathrm{Hom}(X, X) \cong \mathbb{F}$. Otherwise, it is non-split. A category is split if all of its simple objects are.

Project

Our project aims to construct this "pushout" diagram:

$$\begin{tikzcd}
\text{[TY98]: Split TY (AC)} \arrow[r] \arrow[d] & \text{[PSS23]: Non-split TY} \arrow[d, dashed] \\
\text{[Sie00]: Braided Split TY (AC)} \arrow[r, dashed] & \text{Project: Braided Non-split TY over } \mathbb{R} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Strategy, briefly

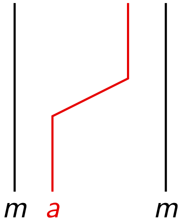
1. Pick "basis" for category: objects are finite direct sums of simple objects; morphisms look like "matrices"
2. Schur's lemma: tells me what morphisms are between simple objects
3. Yoneda lemma: examine morphisms by looking at effects of precomposition, which are *linear maps* between Hom spaces
4. Pick bases for Hom spaces and compute

Classification of Associators

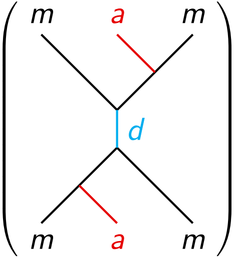
Split Case

Theorem: [Tambara-Yamagami '98] Any split TY category is determined by a triple (G, χ, τ), where G is a finite group, $\chi\colon G \times G \to \mathbb{R}^\times$ is a nondegenerate symmetric bicharacter, and $\tau \in \{\pm 1/\sqrt{|G|}\}$. Associators can be explicitly constructed:

$\alpha_{m,a,m}$ =



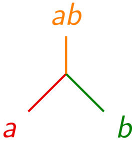
$= \sum_{d\in G} \chi(a,d) \cdot$



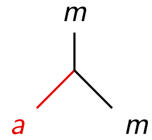
Bases for Hom's

Split Case

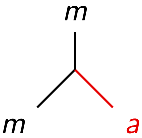
(continued) where trivalent vertices denote a set of fixed basis vectors:



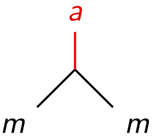
$\in \mathrm{Hom}(a \otimes b, ab)$



$\in \mathrm{Hom}(a \otimes m, m)$



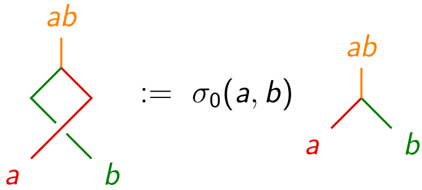
$\in \mathrm{Hom}(m \otimes a, m)$



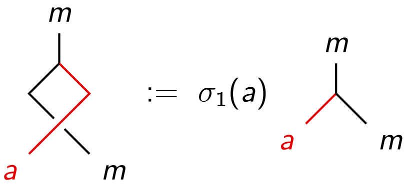
$\in \mathrm{Hom}(m \otimes m, a)$

Braiding Coefficients

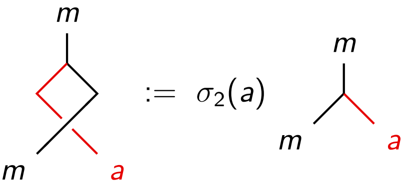
Split Case



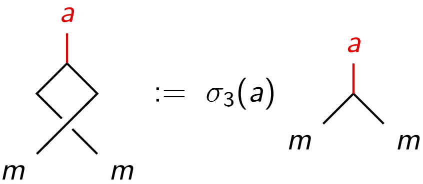
$$:= \sigma_0(a, b)$$



$$:= \sigma_1(a)$$



$$:= \sigma_2(a)$$

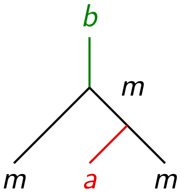


$$:= \sigma_3(a)$$

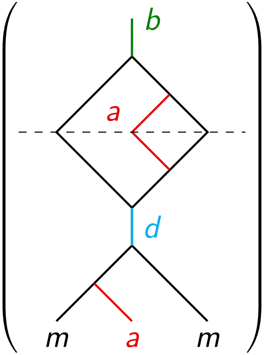
Example Computation

Split Case

(Top path, hexagon 6)



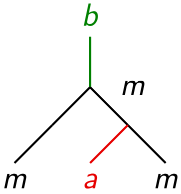
$$\xmapsto{\alpha^*_{m,a,m}} \sum_{d\in A} \chi(a,d)\cdot$$



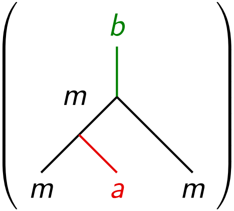
Example Computation

Split Case

(Top path, hexagon 6)

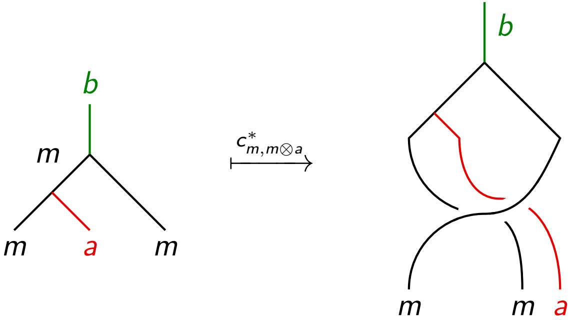


$$\xmapsto{\alpha^*_{m,a,m}} \chi(a, b) \cdot$$



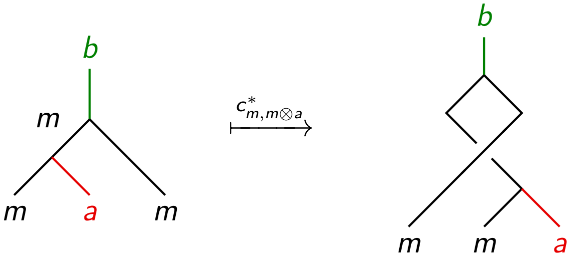
Example Computation

Split Case



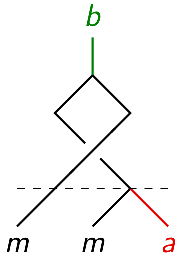
Example Computation

Split Case

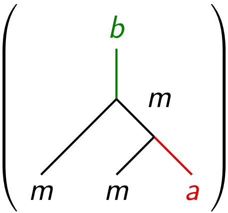


Example Computation

Split Case

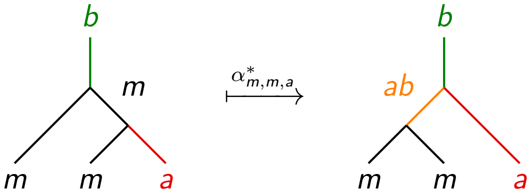


$$= \quad \sigma_3(b) \cdot$$



Example Computation

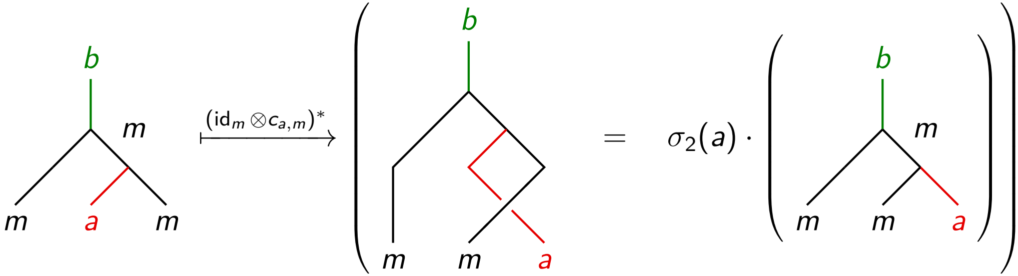
Split Case



Example Computation

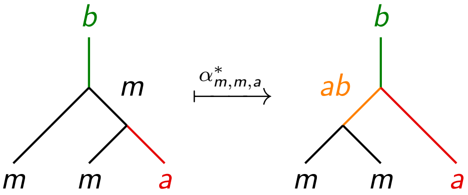
Split Case

(Bottom path, hexagon 6)



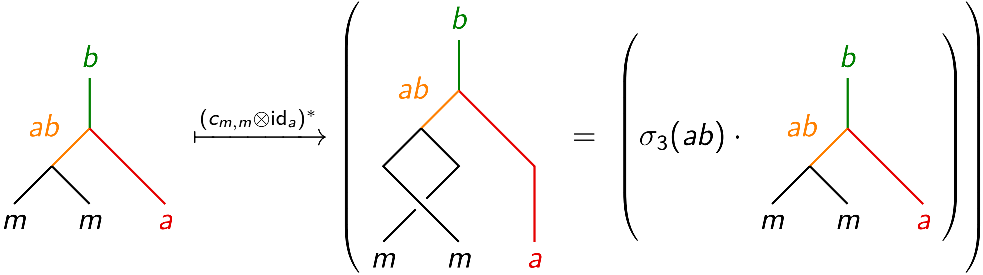
Example Computation

Split Case



Example Computation

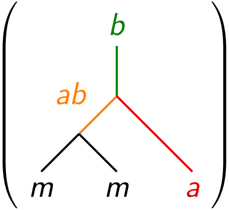
Split Case



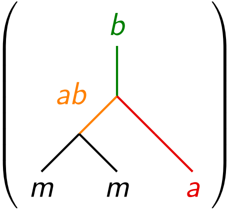
Example Computation - Result

Split Case

$$\sigma_3(b)\chi(a, b)$$



$$= \sigma_3(ba)\sigma_2(a)$$



Linear independence:

$$\sigma_3(b)\chi(a, b) = \sigma_3(ab)\sigma_2(a)$$

Do this fifteen more times to get: (con't on next slide)

Hexagon Equations

Split Case

Hexagon axioms $\iff$ sixteen hexagon equations $\iff$ six reduced equations:

$$\sigma_0(a,b) = \chi(a,b), \tag{1}$$

$$\sigma_1(a)^2 = \chi(a,a), \tag{2}$$

$$\sigma_1(ab) = \sigma_1(a)\sigma_1(b)\chi(a,b), \tag{3}$$

$$\sigma_2(a) = \sigma_1(a), \tag{4}$$

$$\sigma_3(1)^2 = \tau \sum_{c \in G} \sigma_1(c), \tag{5}$$

$$\sigma_3(a) = \sigma_3(1)\sigma_1(a)\chi(a,a). \tag{6}$$

Equation (2) tells us that $\chi(a,a) > 0$ for all $a \in G$, since $\sqrt{-1} \notin \mathbb{R}$.

Classification of Braidings

Split Case

Equation (2) tells us that $\chi(a, a) > 0$ for all $a \in G$, which places a big restriction on χ ([Wall '63]), leading to the following classification.

Theorem: [J.] Any split TY category over $\mathbb{R}$ that admits a braiding is equivalent to $\mathcal{C}(K_4^n, h^{\oplus n}, \tau)$, where K_4 denotes the Klein four-group, h denotes the hyperbolic pairing on K_4, $\tau \in \{\pm {}^1\!/\!_{2^n}\}$, and $n \in \mathbb{N}$.

Classification of Braidings

Split Case

If we also consider the restriction

$$\sigma_3(1)^2 = \tau \sum_{c \in G} \sigma_1(c),$$

we have a classification of the solutions to the hexagon equations (the braidings), as well as a classification up to (monoidal) equivalence:

There are two non-equivalent braidings on $\mathcal{C}(K_4^n, h^{\oplus n}, \tau)$.

Non-Split Cases

For all simples $X \in \mathcal{C}$, $\mathrm{End}(X)$ is not necessarily the ground field $\mathbb{R}$, but we have the following:

- Lemma (Schur): $\mathrm{End}(X)$ is a division algebra over $\mathbb{R}$
- Theorem (Frobenius): Finite dimensional division algebra over $\mathbb{R}$ is $\mathbb{R}, \mathbb{C}$,or $\mathbb{H}$
- Results from [Plavnik-Sanford-Sconce '23]:
  - $\mathrm{End}(g) \cong \mathrm{End}(1_G)$ for all $g \in G$;
  - $\mathrm{End}(1_G)$ is commutative;
  - $\mathrm{End}(1_G) \subseteq Z(\mathrm{End}(m))$

Non-Split Cases

$\mathrm{End}(1_G)$	$\mathrm{End}(m)$	Viable?
$\mathbb{R}$	$\mathbb{R}$	Split
	$\mathbb{H}$	Case 1
	$\mathbb{C}$	Case 2
$\mathbb{C}$	$\mathbb{R}$	$Z(\mathbb{R}) = \mathbb{R}$
	$\mathbb{H}$	$Z(\mathbb{H}) = \mathbb{R}$
	$\mathbb{C}$	Case 3

Case 3 does not admit braidings. We will look at Case 1 for the rest of the talk.

Classification of Associators

Real/Quaternionic Case

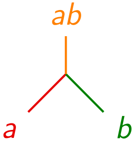
The fact that $Z(\mathbb{H}) = \mathbb{R}$ makes the real/quaternionic case feel very similar to the split case.

Theorem: [Plavnik-Sanford-Sconce '23] All real/quaternionic TY categories arise as $\mathcal{C}_{\mathbb{H}}(G, \chi, \tau)$, where G is a finite group, $\chi : G \times G \to \mathbb{R}^{\times}$ is a nondegenerate symmetric bicharacter, and $\tau \in \{\pm 1/\sqrt{4|G|}\}$.

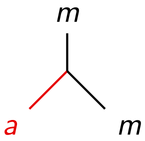
Associators can be constructed using fixed basis vectors: (continued on next slide)

Bases for Hom's

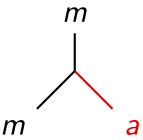
Real/Quaternionic Case



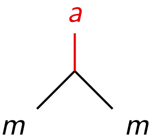
$\in \mathrm{Hom}(a \otimes b, ab)$



$\in \mathrm{Hom}(a \otimes m, m)$



$\in \mathrm{Hom}(m \otimes a, m)$



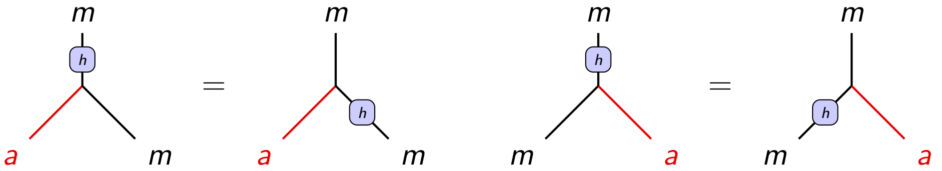
$\in \mathrm{Hom}(m \otimes m, a)$

these have an additional nice property: (continued on next slide)

Bases for Hom's

Real/Quaternionic Case

The chosen basis vectors have the following property: we can pass all $h \in \mathbb{H}$ through vertices:

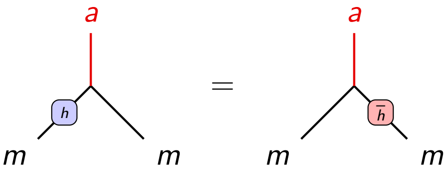


(continued on next slide)

Bases for Hom's

Real/Quaternionic Case

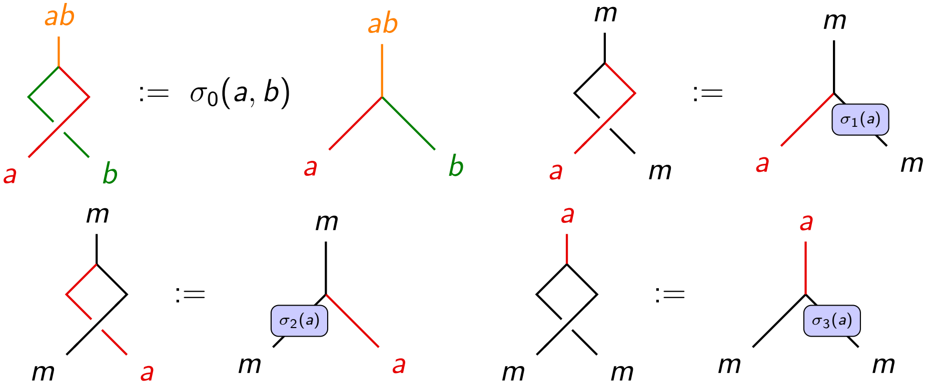
(continued)



. . . but sometimes they get conjugated.

Braiding Coefficients

Real/Quaternionic Case



Lemma

Non-Split Cases

. . . but the σ's actually are all real valued!

Lemma: The function σ_1 is real-valued.

Proof: We want to show that for all $a \in A$ and all $h \in \mathbb{H}$,

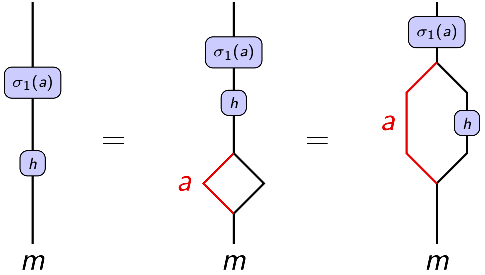
$$\sigma_1(a) \cdot h = h \cdot \sigma_1(a),$$

which would tell us that $\sigma_1(a) \in \Big(Z(\mathbb{H}) = \mathbb{R}\Big)$. (con't on next slide)

Lemma (Proof)

Non-Split Cases

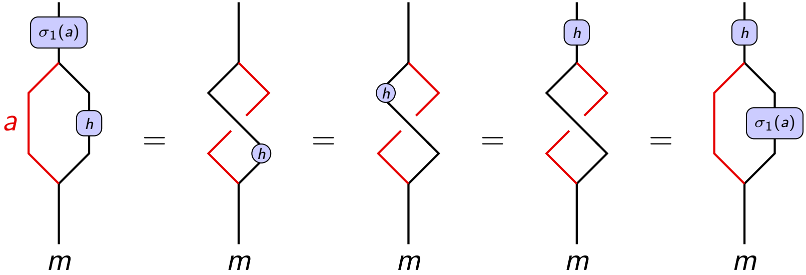
With graphical calculus, we have



Lemma (Proof)

Non-Split Cases

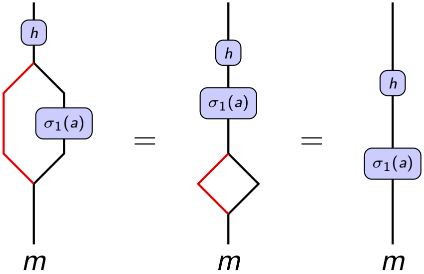
Now, by the definition of σ_1 and naturality, we have



Lemma (Proof)

Non-Split Cases

Pass $\sigma_1(a)$ through the trivalent vertex to get



as desired. $\square$

Hexagon Equations

Real/Quaternionic Case

The reduced hexagon are as follows (note that $(\star)$ is the only one different from the split case):

$$\begin{aligned}
\sigma_0(a,b) &= \chi(a,b), \\
\sigma_1(a)^2 &= \chi(a,a), \\
\sigma_1(ab) &= \sigma_1(a)\sigma_1(b)\chi(a,b), \\
\sigma_2(a) &= \sigma_1(a), \\
\sigma_3(1)^2 &= -2 \cdot \tau \sum_{c \in A} \sigma_1(c), \qquad (\star) \\
\sigma_3(a) &= \sigma_3(1)\sigma_1(a)\chi(a,a).
\end{aligned}$$

Classification of Braidings

Real/Quaternionic Case

The classification is very similar to the split case (differences arise in the actual construction of the braidings).

Theorem: [J.] Any real-quaternionic TY category that admits a braiding is equivalent to $\mathcal{C}_{\mathbb{H}}(K_4^n, h^{\oplus n}, \tau)$, where $\tau \in \{\pm 1/2^{n+1}\}$. There are two non-equivalent braidings on each of these categories.

Future Directions

- Case 2 (Real/Complex): graphical computations produce a set of different equations which involve complex conjugation
- Case 3 (Complex/Complex): does not admit braidings but can admit "G-crossed braidings"

References

[PSS23] Julia Plavnik, Sean Sanford, and Dalton Sconce. *Tambara-Yamagami Categories over the Reals: The Non-Split Case*. 2023. arXiv: `2303.17843 [math.QA]`.

[Sie00] Jacob A. Siehler. *Braided Near-group Categories*. 2000. arXiv: `math/0011037 [math.QA]`.

[TY98] Daisuke Tambara and Shigeru Yamagami. "Tensor categories with fusion rules of self-duality for finite abelian groups". In: *Journal of Algebra* 209.2 (1998), pp. 692–707. DOI: `https://doi.org/10.1006/jabr.1998.7558`.

[Wal63] C. T. C. Wall. "Quadratic forms on finite groups, and related topics". In: *Topology* 2 (1963), pp. 281–298. DOI: `10.1016/0040-9383(63)90012-0`.

Thank you!